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VARIATIONAL FORMULATIONS OF BOUNDARY-VALUE PROBLEMS IN MATERIAL FAILURE

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In [1] we considered the formulation of boundary-value problems in the deformation of materials which possess the property of localizing slip. In the present article we consider variational formulations for the general case of material failure, when there are severe discontinuities both in the tangential and in the normal components of the displacements.

1. We shall confine our attention to the case of plane deformation or a plane stressed state. We introduce the cartesian coordinates Ox_1x_2. We denote by S the deformed region bounded by the contour Γ. Suppose that for specified load parameters the region is divided by a line of severe discontinuity of the displacements. Hereafter, it will be sufficient to confine our attention to the case of a single line. The results remain valid for the case of several lines. We assume first of all that the trajectory of the distribution of the lines is known either from experimental data or from symmetry conditions, or that it is given on the basis of additional concentrations. For one rather broad class of models the boundary-value problem of the propagation of the discontinuity lines and the deformation of the material outside the lines can be reduced to the determination of the stationary values of certain functionals on the class of discontinuous functions. The functionals must depend both on the behavior of the functions in the region of smoothness and on the value of the discontinuities of these functions. Suppose that in S there are given certain fields of stresses σ_{ij}, displacements u_k, and variables $\lambda_r (i, j, k = 1, 2, r = 1, 2, \ldots)$. We define the functional

$$W = \int\limits_{S^+} F\left(u_i^+, u_{i,i}^+, \gamma^+, \sigma_{ij}^+, \sigma_{ij,k}^+, \lambda_r^+, \lambda_{r,k}^+, x_k\right) dx_1 dx_2 + \int\limits_{S^-} F\left(u_i^-, \ldots\right) dx_1 dx_2 + \int\limits_L U dl - \int\limits_\Gamma \Pi dl, \qquad (1.1)$$

where $\lambda = u_{1,2} + u_{2,1}$; $\sigma_{12} = \sigma_{21}$; U, Π are functions defined on the line of discontinuity L and the external boundary. Here and hereafter, a comma before a subscript denotes differentiation with respect to the appropriate coordinate, and the superscripts +, –, indicate the notation in the regions to the right and left of L. We assume that on real solutions the functional is stationary. We consider the formulation when all equations are completely determined by the variational principle, and consequently there are no other limitations within S^+, S^-. The necessary condition for an extremum leads to the following system of Euler–Ostrogradsky equations:

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$$F_{u_1}-\frac{\partial}{\partial x_1}\{F_{u_{1,1}}\}-\frac{\partial}{\partial x_2}\{F_\gamma\}=0,\quad F_{u_2}-\frac{\partial}{\partial x_1}\{F_\gamma\}-\frac{\partial}{\partial x_2}\{F_{u_{2,2}}\}=0; \tag{1.2}$$

$$F_{\sigma_{11}}-\frac{\partial}{\partial x_1}\{F_{\sigma_{11,1}}\}-\frac{\partial}{\partial x_2}\{F_{\sigma_{11,2}}\}=0, \tag{1.3}$$

$$F_{\sigma_{22}}-\frac{\partial}{\partial x_1}\{F_{\sigma_{22,1}}\}-\frac{\partial}{\partial x_2}\{F_{\sigma_{22,2}}\}=0,\quad F_{\sigma_{12}}-\frac{\partial}{\partial x_1}\{F_{\sigma_{12,1}}\}-\frac{\partial}{\partial x_2}\{F_{\sigma_{12,2}}\}=0;$$

$$F_{\lambda_r}-\frac{\partial}{\partial x_1}\{F_{\lambda_{r,1}}\}-\frac{\partial}{\partial x_2}\{F_{\lambda_{r,2}}\}=0; \tag{1.4}$$

the condition on the line of possible discontinuity

$$\begin{gathered}A_1^-\delta u_1^-+A_2^-\delta u_2^-+B_{11}^-\delta\sigma_{11}^-+B_{22}^-\delta\sigma_{22}^-+B_{12}^-\delta\sigma_{12}^-+\Lambda_r^-\delta\lambda_r^- -\\ -\left[A_1^+\delta u_1^++A_2^+\delta u_2^++B_{11}^+\delta\sigma_{11}^++B_{22}^+\delta\sigma_{22}^++B_{12}^+\delta\sigma_{12}^++\Lambda_r^+\delta\lambda_r^+\right]+\delta U=0\end{gathered} \tag{1.5}$$

and the condition on the external boundary Γ

$$\begin{gathered}\left(F_{u_{1,1}}\cos\psi+F_\gamma\sin\psi\right)\delta u_1+\left(F_\gamma\cos\psi+F_{u_{2,2}}\sin\psi\right)\delta u_2+\\ +\left(F_{\sigma_{11,1}}\cos\psi+F_{\sigma_{11,2}}\sin\psi\right)\delta\sigma_{11}+\left(F_{\sigma_{22,1}}\cos\psi+F_{\sigma_{22,2}}\sin\psi\right)\delta\sigma_{22}+\\ +\left(F_{\sigma_{12,1}}\cos\psi+F_{\sigma_{12,2}}\sin\psi\right)\delta\sigma_{12}+\left(F_{\lambda_{r,1}}\cos\psi+F_{\lambda_{r,2}}\sin\psi\right)\delta\lambda_r=\delta\Pi.\end{gathered} \tag{1.6}$$

The subscripts of the function F denote partial derivatives; the braces denote total derivatives; ψ is the angle between the external normal to S and the axis Ox_1; α is the angle between the normal $\bar{n}$ to L and the axis Ox_1 (the normal is external with respect to the region S^-). Furthermore, we use the following notation in (1.5):

$$\begin{gathered}A_1=F_{u_{1,1}}\cos\alpha+F_\gamma\sin\alpha,\quad A_2=F_\gamma\cos\alpha+F_{u_{2,2}}\sin\alpha,\\ B_{11}=F_{\sigma_{11,1}}\cos\alpha+F_{\sigma_{11,2}}\sin\alpha,\quad B_{22}=F_{\sigma_{22,1}}\cos\alpha+F_{\sigma_{22,2}}\sin\alpha,\\ B_{12}=F_{\sigma_{12,1}}\cos\alpha+F_{\sigma_{12,2}}\sin\alpha,\quad \Lambda_r=F_{\lambda_{r,1}}\cos\alpha+F_{\lambda_{r,2}}\sin\alpha.\end{gathered} \tag{1.7}$$

The resulting equations, boundary conditions, and conditions on the line of discontinuity are written in general form and outline a specific class of deformation and failure models. Our further investigation may be carried out in several possible ways. First of all, we note that the system (1.2)-(1.4) must be equivalent to the equations of equilibrium and state. Furthermore, all the equations must be invariant with respect to translation and rotation of the coordinate system. The conjugacy conditions on L and the boundary conditions must have a specific mechanical meaning. The general restriction on the functional may be investigated on the basis of any of these three requirements. The simplest way is to analyze the mechanical meaning of the conditions on L and Γ.

We introduce on L the following notation:

$$\begin{gathered}s_i=u_i^++u_i^-,\ R_i=u_i^+-u_i^-,\ R_n=\cos\alpha\,R_1+\sin\alpha\,R_2,\\ R_m=-\sin\alpha\,R_1+\cos\alpha\,R_2,\quad \Sigma_1=\sigma_{11}\cos\alpha+\sigma_{12}\sin\alpha,\\ \Sigma_2=\sigma_{12}\cos\alpha+\sigma_{22}\sin\alpha,\quad \Sigma_n=\cos\alpha\Sigma_1+\sin\alpha\Sigma_2,\\ \Sigma_m=-\sin\alpha\Sigma_1+\cos\alpha\Sigma_2.\end{gathered} \tag{1.8}$$

The quantities R_i, R_n, R_m are the projections of the discontinuity in the displacements onto the axes Ox_i and the directions $\bar{n}=\{\cos\alpha,\ \sin\alpha\}$, $\bar{m}=\{-\sin\alpha,\ \cos\alpha\}$; Σ_i, Σ_n, Σ_m represent the same projections of the stress vector on an area tangent to L.

We assumed earlier that the functional is stationary on the fields of displacements, stresses, and variables λ_r, which are independent only within the regions S^+, S^-. On the contours Γ and L we admit the possibility of constraints which may or may not be "cooled" by the variational principle. In particular, on L we shall always assume two stress continuity conditions: $\Sigma_n^+=\Sigma_n^-$, $\Sigma_m^+=\Sigma_m^-$, or

$$\begin{gathered}(M\sigma_{11}\cos\alpha+M\sigma_{12}\sin\alpha)^+=(M\sigma_{11}\cos\alpha+M\sigma_{12}\sin\alpha)^-,\\ (M\sigma_{12}\cos\alpha+M\sigma_{22}\sin\alpha)^+=(M\sigma_{12}\cos\alpha+M\sigma_{22}\sin\alpha)^-,\end{gathered} \tag{1.9}$$

where M is a constant. The condition (1.5), taking account of (1.8), (1.9), can be represented in the form

$$\begin{aligned}(1.10)\quad & \delta\sigma_{12}^{-}\left[-\operatorname{tg}\alpha B_{11}^{-}-\operatorname{ctg}\alpha B_{22}^{-}+B_{12}^{-}\right]-\delta\sigma_{12}^{+}\left[-\operatorname{tg}\alpha B_{11}^{+}-\operatorname{ctg}\alpha B_{22}^{+}+B_{12}^{+}\right]+ \\ & +\delta\Sigma_1\frac{B_{11}^{-}-B_{11}^{+}}{\cos\alpha}+\delta\Sigma_2\frac{B_{22}^{-}-B_{22}^{+}}{\sin\alpha}+\delta s_1\frac{A_1^{-}-A_1^{+}}{2}+\delta s_2\frac{A_2^{-}-A_2^{+}}{2}+ \\ & +\delta R_n\left[-\frac{A_1^{+}+A_1^{-}}{2}\cos\alpha-\frac{A_2^{+}+A_2^{-}}{2}\sin\alpha\right]+ \\ & +\delta R_m\left[\frac{A_1^{+}+A_1^{-}}{2}\sin\alpha-\frac{A_2^{+}+A_2^{-}}{2}\cos\alpha\right]+\left[\Lambda_r^{-}\delta\lambda_r^{-}-\Lambda_r^{+}\delta\lambda_r^{+}\right]+\delta U=0.\end{aligned} \tag{1.10}$$

The function U must be invariant with respect to spatial displacements ($U_{s_i}\equiv 0$). Then from (1.10), (1.7) we obtain two continuity conditions: $A_i^{+}=A_i^{-}$ or

$$\begin{aligned}&\left(F_{u_{1,1}}\cos\alpha+F_\gamma\sin\alpha\right)^{+}=\left(F_{u_{1,1}}\cos\alpha+F_\gamma\sin\alpha\right)^{-},\\ &\left(F_\gamma\cos\alpha+F_{u_{2,2}}\sin\alpha\right)^{+}=\left(F_\gamma\cos\alpha+F_{u_{2,2}}\sin\alpha\right)^{-}.\end{aligned} \tag{1.11}$$

Thus, from the assumption that a variational formulation exists and that the function defined on the line of discontinuity is invariant with respect to spatial displacements, we obtain two continuity (conservation) conditions on this line. This result is not accidental; it is a special case of known fundamental connections between the properties of invariance and the laws of conservation.

It is natural to assume that the function U may depend only on continuous stress components. Therefore, in the general case the brackets that are multiplied by the variations $\delta\sigma_{12}$ must vanish. Unlike (1.1), this last requirement leads to constraints on the variables only on one side of L:

$$F_{\sigma_{11,1}}-F_{\sigma_{12,2}}\equiv 0,\quad F_{\sigma_{22,2}}-F_{\sigma_{12,1}}\equiv 0,\quad F_{\sigma_{11,2}}\equiv 0,\quad F_{\sigma_{22,1}}\equiv 0.$$

Consequently, the generating function F can depend on the derivatives of the stresses only through the combinations

$$p_1=\sigma_{11,1}+\sigma_{12,2},\quad p_2=\sigma_{12,1}+\sigma_{22,2},\quad F=F(\ldots p_1,\,p_2\ldots). \tag{1.12}$$

The validity of the representation (1.12) follows also from the fact that on the boundary Γ there may be specified some information concerning the stresses defined only on the areas tangent to Γ. The necessity of (1.12) for interior points of S^{+}, S^{-} can be shown by starting with the requirements of invariance.

The results enable us to simplify somewhat the system (1.3) and the conditions (1.5), (1.6):

$$\frac{\partial}{\partial x_1}F_{p_1}=F_{\sigma_{11}},\quad \frac{\partial}{\partial x_2}F_{p_2}=F_{\sigma_{22}},\quad \frac{\partial}{\partial x_1}F_{p_2}+\frac{\partial}{\partial x_2}F_{p_1}=F_{\sigma_{12}}; \tag{1.13}$$

$$\begin{aligned}&\delta\Sigma_n\left[\left(F_{p_1^{+}}-F_{p_1^{-}}\right)\cos\alpha+\left(F_{p_2^{+}}-F_{p_2^{-}}\right)\sin\alpha\right]+\\ &+\delta\Sigma_m\left[-\left(F_{p_1^{+}}-F_{p_1^{-}}\right)\sin\alpha+\left(F_{p_2^{+}}-F_{p_2^{-}}\right)\cos\alpha\right]+\\ &+A_n\delta R_n+A_m\delta R_m+\Lambda_r^{+}\delta\lambda_r^{+}-\Lambda_r^{-}\delta\lambda_r^{-}=\delta U,\end{aligned} \tag{1.14}$$

where

$$\begin{aligned}&A_n=A_1\cos\alpha+A_2\sin\alpha;\quad A_m=-A_1\sin\alpha+A_2\cos\alpha;\\ &\left(F_{u_{1,1}}\cos\psi+F_\gamma\sin\psi\right)\delta u_1+\left(F_\gamma\cos\psi+F_{u_{2,2}}\sin\psi\right)\delta u_2+\\ &+F_{p_1}(\delta\sigma_{11}\cos\psi+\delta\sigma_{12}\sin\psi)+F_{p_2}(\delta\sigma_{12}\cos\psi+\delta\sigma_{22}\sin\psi)+\\ &+\left(F_{\lambda_{r,1}}\cos\psi+F_{\lambda_{r,2}}\sin\psi\right)\delta\lambda_r=\delta H.\end{aligned} \tag{1.15}$$

We shall confine our attention to the classical case in which the state of an elementary volume of the medium is completely characterized by the stress and deformation tensors. From this it follows that on the external boundary, information can be specified only concerning the displacements or stresses. Therefore, as is shown by formula (1.15), the derivatives F_{p_i} must be expressed in terms of the displacements

$$F_{p_1} = Ku_1, \quad F_{p_2} = Ku_2, \tag{1.16}$$

where K is a given constant.

We noted earlier that on L the two stress continuity conditions (1.9) and the continuity condition (1.11) must be satisfied. The formulas (1.11) should not specify any new conditions in relation to (1.9). This last requirement will be our basis for the construction of the functionals.

If a functional depends on the derivatives $\lambda_{r,k}$, then, as is shown by condition (1.15), the variables λ_r must reduce either to displacements or to stresses. The introduction of such variables leads to a problem in a space of larger dimension, where equations of the type $\lambda_r = \sigma_{ij}$, u_k must be satisfied on the solution. The generalizations to this case are obvious and will not be considered further. If the functional depends on the variables λ_r themselves, then Eqs. (1.3), (1.4) will describe in the general case some plastic state of the medium. On the other hand, the diagram of stresses versus discontinuities of the displacements, which is determined by the function U, must, by virtue of its mechanical meaning, have a descending branch. Therefore, if we take account of possible plastic flow outside the line of discontinuity, we must also take account of the unloading effects which take place as a result of the development of the line. In the present study we shall not consider this problem, and therefore the case in which the functional depends on λ_r will be excluded.

The continuity conditions (1.9), (1.11) show that the system (1.2) must have the meaning of equilibrium equations. We assume that the bulk forces are independent of the stressed state ($F_{u_k \sigma_{ij}} \equiv 0$). From this and (1.16) we obtain the representation

$$F = K(u_1 p_1 + u_2 p_2) + X(u_1, u_2) + F^0(\sigma_{ij}, \varepsilon_{ij}, x_k), \tag{1.17}$$

where $\varepsilon_{ii} = u_{i,i}$, $\varepsilon_{12} = \gamma$. The last statement enables us to simplify the equations (1.13):

$$F^0_{\sigma_{11}}(\sigma_{ij}, \varepsilon_{ij}, x_k) = K\varepsilon_{11}, \quad F^0_{\sigma_{22}} = K\varepsilon_{22}, \quad F^0_{\sigma_{12}} = K\varepsilon_{12}. \tag{1.18}$$

The equations (1.18) constitute an algebraic system with respect to the stress or deformations. Suppose that the solution of the system has the form

$$\sigma_{ij} = \sigma_{ij}(\varepsilon_{hh}, x_k), \quad \varepsilon_{ij} = \varepsilon_{ij}(\sigma_{hh}, x_k) \quad (h = 1, 2). \tag{1.19}$$

(The function F^0 must be such that the solution is independent of the parameter K.)

The requirement that the continuity conditions (1.9), (1.11) coincide imposes a restriction on the admissible class of functions F^0. The minimal restriction on F^0 is that (1.9), (1.11) coincide on the solution (1.19). From a comparison of (1.9) with (1.11) it follows that on the solution (1.19) the equations

$$F^0_{\varepsilon_{11}}(\sigma_{ij}, \varepsilon_{ij}, x_k) = M\sigma_{11}, \quad F^0_{\varepsilon_{22}} = M\sigma_{22}, \quad F^0_{\varepsilon_{12}} = M\sigma_{12} \tag{1.20}$$

must be satisfied. Hereafter the equations (1.20) will be regarded as an algebraic system with respect to the stresses or deformations. Then the function F^0 must be such that the solutions of the systems (1.18), (1.20) will coincide with each other and with the given solution (1.19).

The methods for constructing the function F^0 can be most conveniently illustrated by using a simpler one-dimensional situation. In the one-dimensional case the problem reduces to the following. It is required to find a function of two variables $z = f(x, y)$ such that on a given curve $y = y(x)$, $x = x(y)$

$$f_x(x, y) \equiv Ky, \quad f_y(x, y) \equiv Mx.$$

We shall try to find a solution in the form of a sum $f = \Phi(x, y) + \Psi(x, y)$, where $\Phi(x, y)$ is the general solution of the homogeneous problem; for $y = y(x)$ $\Phi_x \equiv 0$, $\Phi_y \equiv 0$, and the function

Ψ is the particular solution of the inhomogeneous problem. The function Φ defines in three-dimensional space a surface with normal $\bar{n}$. Obviously, on the curve $y = y(x)$ $\Phi \equiv \text{const}$, $\bar{n} = \{0, 0, 1\}$. Thus, the surface $z = \Phi(x, y)$ must be tangent to the plane $z = \text{const}$ along the curve $y = y(x)$. The particular solution will be sought in the form

$$\Psi = \Psi_1(x) + \Psi_2(y) + \xi xy,$$

where $\xi = \text{const}$.

Then $f(x, y) = (K - \xi) \int y(x)\, dx + (M - \xi) \int x(y)\, dy + \xi xy + \Phi(x, y)$.

The constructions considered above can be generalized to a multidimensional problem. Suppose that $\Phi(\varepsilon_{ij}, \sigma_{ij}, x_k)$ is the general solution of the homogeneous problem, i.e., on the functions (1.19) $\Phi_{\varepsilon_{ij}} \equiv 0$, $\Phi_{\sigma_{ij}} \equiv 0$. One of the constructions of Φ is the following:

$$\Phi = \Phi[\sigma_{ij} - \sigma_{ij}(\varepsilon_{kh}, x_k), \varepsilon_{ij} - \varepsilon_{ij}(\sigma_{kh}, x_k)],$$

where Φ is an arbitrary function which is equal to zero, together with its first derivatives, when the arguments are zero. The particular solution of the inhomogeneous case can be sought in the form

$$\Psi = \Psi_1(\varepsilon_{ij}, x_k) + \Psi_2(\sigma_{ij}, x_k) + \xi(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{12}\varepsilon_{12}),$$

where the functions Ψ_i are solutions of the differential equations

$$\begin{gathered} \Psi_{1,\varepsilon_{ij}}(\varepsilon_{kh}, x_k) = (M - \xi)\, \sigma_{ij}(\varepsilon_{kh}, x_k), \\ \Psi_{2,\sigma_{ij}}(\sigma_{kh}, x_k) = (K - \xi)\, \varepsilon_{ij}(\sigma_{kh}, x_k). \end{gathered} \tag{1.21}$$

Let us consider in more detail the case of a linearly elastic body:

$$\begin{gathered} \sigma_{11} = A\varepsilon_{11} + B\varepsilon_{22},\ \sigma_{22} = B\varepsilon_{11} + A\varepsilon_{22},\ \sigma_{12} = C\varepsilon_{12}, \\ \varepsilon_{11} = a\sigma_{11} + b\sigma_{22},\ \varepsilon_{22} = b\sigma_{11} + a\sigma_{22},\ \varepsilon_{12} = c\sigma_{12}, \end{gathered}$$

where $A = 2\mu(1 - \nu)/(1 - 2\nu)$, $B = 2\mu\nu/(1 - 2\nu)$, $C = \mu$ for plane deformation; $A = \lambda^* + 2\mu$, $B = \lambda^*$, $C = \mu$, $\lambda^* = E\nu/(1 - \nu^2)$ for a plane stressed state; $a = A/(A^2 - B^2)$; $b = -B/(A^2 - B^2)$; $c = 1/C$; μ, ν, E are elastic constants. The solution of the equations (1.21) has the form

$$\begin{gathered} \Psi_1 = (M - \xi)\left[\frac{A}{2}\left(\varepsilon_{11}^2 + \varepsilon_{22}^2\right) + B\varepsilon_{11}\varepsilon_{22} + \frac{C}{2}\varepsilon_{12}^2\right], \\ \Psi_2 = (K - \xi)\left[\frac{a}{2}\left(\sigma_{11}^2 + \sigma_{22}^2\right) + b\sigma_{11}\sigma_{22} + \frac{c}{2}\sigma_{12}^2\right]. \end{gathered}$$

Now let us consider the restrictions on the functionals that are imposed by the system (1.2). We assume that the bulk forces are independent of the displacements and consequently the function X in the representation (1.17) is linear ($X = Y_1u_1 + Y_2u_2$). The function F is such that the system (1.2), taking account of (1.3), is transformed to the form

$$(K - M)p_1 + Y_1 = 0,\ (K - M)p_2 + Y_2 = 0.$$

From this we obtain a necessary condition for the solvability of the problem: $K \neq M$, and the mechanical meaning of the coefficients: $Y_i = (K - M)X_i$, where X_i are the components of the volumetric forces.

Thus, the failure problems under consideration (in particular the problems relating to deformation without discontinuities) admit of infinitely many variational formulations corresponding to the various generating functions F:

$$\begin{gathered} F = K(u_1p_1 + u_2p_2) + Y_1u_1 + Y_2u_2 + \Phi(\varepsilon_{ij}, \sigma_{ij}, x_k) + \Psi_1(\varepsilon_{ij}, x_k) + \\ + \Psi_2(\sigma_{ij}, x_k) + \xi(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{12}\varepsilon_{12}). \end{gathered}$$

We shall indicate the most important special cases:

1) Let $\xi = 0$, $K = 0$, $M = 1$, $\Phi \equiv 0$. Then the variational principle reduces to the generalized principle of possible displacements. This variant, as well as the examples of the solution of the boundary-value problems of the stable and unstable development of cracks were considered [2, 3].

2) If $\xi = 0$, $K = 1$, $M = 0$, $\Phi \equiv 0$, the principle reduces to the generalized Castigliano principle.

3) The case $\xi = 1$, $K = 0$, $M = 1$, $\Phi \equiv 0$ corresponds to Reissner's principle.

2. We consider the most important types of boundary conditions. The conditions on the external boundary (1.15), taking account of the equations (1.18), are transformed to the form

$$M\Sigma_1\delta u_1 + M\Sigma_2\delta u_2 + Ku_1\delta\Sigma_1 + Ku_2\delta\Sigma_2 = \delta\Pi. \tag{2.1}$$

Obviously the function Π can depend only the the arguments u_i, Σ_i, x_k. Suppose that $\Pi = \Sigma_1 Ku_1^0 + \Sigma_2 Ku_2^0$ and on the boundary only the stresses ($u_i = u_i^0$) vary. The indicated form of Π describes the given boundary displacements. If on the boundary only the displacements ($\Sigma_i = \Sigma_i^0$) vary, then $\Pi = M\Sigma_1^0 u_1 + M\Sigma_2^0 u_2$. This case corresponds to given boundary stresses.

Now let us assume that on the boundary we admit the possibility of variations both of the displacements and of the stresses. Without loss of generality, we can assume that $\Pi = g(u_i, x_k) + \varphi(\Sigma_i, x_k)$. From (2.1) it follows that

$$M\Sigma_1 = g_{u_1}(u_i, x_k),\ M\Sigma_2 = g_{u_2}(u_i, x_k); \tag{2.2}$$

$$Ku_1 = \varphi_{\Sigma_1}(\Sigma_i, x_k),\ Ku_2 = \varphi_{\Sigma_2}(\Sigma_i, x_k). \tag{2.3}$$

This variant means that on the boundary we are given the stresses as functions of the displacements or the displacements as functions of the stresses. The conditions (2.2), (2.3) express the same mechanical relationship. Therefore, the functions g, φ must be interrelated.

One fact is particularly worth noting. If we formally set $M = K$, then the left side of (2.1) can be represented as a total differential. Therefore, setting $\Pi = M(\Sigma_1 u_1 + \Sigma_2 u_2)$, we can make sure that the boundary conditions are identically satisfied. Consequently, in this case the information concerning the boundary conditions is excluded from the functional, and the variational problem becomes, generally speaking, indeterminate. Such a situation is precluded by the restriction $M \neq K$ obtained above from other considerations.

Now let us consider the question of the conditions on the line of discontinuity. If we take account of the fact that the systems (1.18) and (1.20) are equivalent, we can transform the conditions (1.14) to the form

$$M\Sigma_n\delta R_n + M\Sigma_m\delta R_m + KR_n\delta\Sigma_n + KR_m\delta\Sigma_m = \delta U. \tag{2.4}$$

On the line of possibility discontinuity L the condition that the regions S^-, S^+ do not overlap must be satisfied. For the numerical solution of specific problems, this condition can be most conveniently satisfied by setting U sufficiently high if the test values of R_n, R_m yield overlapping regions [2]. Suppose that the functional depends only the displacements and there are no other kinematic restrictions on the line of discontinuity. Then it follows from (2.4) that

$$A_n = U_{R_n}(R_n, R_m, x_k),\ A_m = U_{R_m}(R_n, R_m, x_k). \tag{2.5}$$

The last conditions have the meaning of relations connecting the normal and tangential stresses acting on the line to the discontinuities in the displacements. If on L the normal component of the discontinuity is explicitly related to the tangential component

$$f(R_n, R_m) = 0, \tag{2.6}$$

then the second condition has the form

$$f_{R_m}(U_{R_n} - A_n) - f_{R_n}(U_{R_m} - A_m) = 0.$$

By analogy with the definition of dilatancy as a material's property of changing its volume when slip occurs, the condition (2.6) can be called the condition of localized dilatancy. If the functional depends only on the stresses and there are no other restrictions on the line of discontinuity, then on L the following conditions are satisfied:

$$\begin{gathered} -\left(F_{p_1^+} - F_{p_1^-}\right)\sin\alpha + \left(F_{p_2^+} - F_{p_2^-}\right)\cos\alpha = U_{\Sigma_m}, \\ \left(F_{p_1^+} - F_{p_1^-}\right)\cos\alpha + \left(F_{p_2^+} - F_{p_2^-}\right)\sin\alpha = U_{\Sigma_n}; \end{gathered} \tag{2.7}$$

these have the same meaning as (2.5). In the general case, when the functional depends both on the stresses and on the displacements, the problem satisfies the conditions (2.5), (2.7), and the function U must be such that its partial derivatives will describe the same connection between the stresses and the displacement discontinuities.

Starting from (2.1), (2.4), we can consider more complicated boundary conditions and properties of the material on the line of discontinuity.

3. Earlier, we assumed that the lines of possible discontinuity are known in advance. In the exact formulation, the lines must be determined in the process solving the problem. If the line of discontinuity is not known in advance, then W is a functional with respect to the curve L, the stresses, the displacements, and the variables λ_r. It is natural to strengthen the variational principle we have adopted and consider those lines of discontinuity which give the functional the deepest minimum (or stationary value). Suppose that $x_1 = x_1(t)$, $x_2 = x_2(t)$, $t \in [t_1, t_2]$ are parametric equations of the curve L,, $s = \sqrt{(x_1')^2 + (x_2')^2}$. The variational line of possible discontinuity will be considered close to the initial line in the sense of first-order proximity (δx_i, $\delta x_i' \ll 1$). The necessary condition for stationarity of the functional leads to the previous boundary conditions, relations on the line of possible discontinuity, and equations in the regions of smoothness. Additional relations will be obtained for the lines of possible discontinuity. Omitting the intermediate calculations, we give the final results. If the functional is independent of the stresses and the variations δR_m, δR_n are independent or are related by a condition independent of x_i, then

$$\begin{gathered} (U\sin\alpha)' + U_{x_1} + (\varkappa_\alpha \cos\alpha)' + \varkappa_1 - [F]\cos\alpha = 0, \\ -(U\cos\alpha)' + U_{x_2} + (\varkappa_\alpha \sin\alpha)' + \varkappa_2 - [F]\sin\alpha = 0, \end{gathered} \tag{3.1}$$

where the prime denotes the operator d/sdt; $[F] = F^+ - F^-$;

$$\varkappa_1 = A_1 R_{1,1} + A_2 R_{2,1};\ \varkappa_2 = A_1 R_{1,2} + A_2 R_{2,2};\ \varkappa_\alpha = A_1 R_2 - A_2 R_1.$$

The equations (3.1) are obtained on the assumption that the two variations δx_i are independent. Consequently, (3.1) contains information on the variation of L along itself as well. It can be shown that the sum of these equations, multiplied by ($-\sin\alpha$) and $\cos\alpha$, does in fact lead to an identity. As the independent equation, it is natural to take a linear combination of the equations (3.1) with the coefficients $\cos\alpha$, $\sin\alpha$:

$$U\alpha' + U_{x_1}\cos\alpha + U_{x_2}\sin\alpha + (\varkappa_\alpha)' + \varkappa_1\cos\alpha + \varkappa_2\sin\alpha - [F] = 0. \tag{3.2}$$

In the special case in which there are only tangential discontinuities in displacement, (3.2) becomes the corresponding equation of [1]. It should be noted that for central forces of interaction between the two sides of the discontinuity (the stress vector is directed along the displacement discontinuity vector) $U = U\left(\sqrt{R_n^2 + R_m^2}, x_k\right)$, $\varkappa_\alpha \equiv 0$, and Eq. (3.2) is simplified.

If the functional is independent of the derivatives of the displacements, then the equation of the "extremal" line of discontinuity reduces to the following:

$$U\alpha' + U_{x_1}\cos\alpha + U_{x_2}\sin\alpha - [M'_\alpha] - [M_1]\cos\alpha - [M_2]\sin\alpha - [F] = 0,$$

$$\text{where } M_i = -F_{p_1}(\sigma_{11,i}\cos\alpha + \sigma_{12,i}\sin\alpha) - F_{p_2}(\sigma_{12,i}\cos\alpha + \sigma_{22,i}\sin\alpha);$$

$$M_\alpha = -F_{p_1}[-(\sigma_{11} - \sigma_{22})\sin\alpha + 2\sigma_{12}\cos\alpha] +$$
$$+ F_{p_2}[(\sigma_{11} - \sigma_{22})\cos\alpha + 2\sigma_{12}\sin\alpha].$$

In the general case, when the functional is dependent both on the displacements and on the stresses:

$$U\alpha' + U_{x_1}\cos\alpha + U_{x_2}\sin\alpha + (\varkappa_\alpha)' + (\varkappa_1\cos\alpha + \varkappa_2\sin\alpha) -$$
$$- [M'_\alpha] - [M_1]\cos\alpha - [M_2]\sin\alpha - [F] = 0.$$

Let us now consider the question of the conditions for the emergence of the line of discontinuity to the external boundary Γ. For definiteness, we shall consider the boundary condition at the point $t = t_2$. Since $x_i(t_2) \in \Gamma$, it follows that the variations δx_i at this point are interrelated: $\delta x_1 = \delta H \sin\psi$, $\delta x_2 = -\delta H \cos\psi$, where δH is a parameter. If the position of the end of the line of discontinuity is known (for example, $x_i(t_2)$ coincides either with a point of discontinuity of the boundary displacements or stresses or with a point at which the type of boundary condition changes), then $\delta H = 0$ and the boundary condition has the form $x_i(t_2) = x_i^0$, where the x_i^0 are given. This condition satisfies the requirement that W be stationary. If the position of the end of the line of discontinuity is unknown, then $\delta H \neq 0$ and the stationarity requirement leads to the following boundary condition:

$$-U\cos(\psi - \alpha) - \varkappa_\alpha \sin(\psi - \alpha) + [M_\alpha]\sin(\psi - \alpha) - Q = 0,$$
$$\text{where } Q = \Pi^+(u_i^+, \Sigma_i^+, x_k) - \Pi^-(u_i^-, \Sigma_i^-, x_k).$$

Thus, the strengthened variational principle enables us to determine both the discontinuous fields of displacements and stresses and the positions of the lines of discontinuity. However, in taking this approach we do not take account of the history of the loads imposed on the material, and in the general case the "extremal" lines of discontinuity can serve only as estimates of the real deformation processes. (Taking account of the loading history leads to a prohibition of the variations δx_i on those segments of L on which the discontinuity has already taken place).

For specified loading parameters, the discontinuities are not realized and the stationary value of the functional (1.1) will be attained on the class of smooth functions, even though we search for it in the class of discontinuous functions. This result has some features in common with the results obtained in [4]. Earlier, we considered the conditions for stationarity of the complete functional (1.1) only in the space of stresses and displacements. We can consider different transformations of the variational problems by the methods developed in [5, 6]. Further references to studies on the use of variational methods in problems of material failure and model representations are contained in [1, 2].

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